A reliability model for option theory

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Abstract

This paper proposes and explores an extension of the usual h-out-of-n systems, where the components of the system are assumed to play different roles in determining its failure. The theoretical reliability framework is actually adopted for the development of a barrier basket options model. In particular, an option is presented as a coherent system whose components are the assets of the basket. The reliability function of the option – which gives insights on its risk profile – is evaluated by the computation of the signature of the system. Some intuitive financial results serve as illustrations of the theoretical framework.

Keywords: h-out-of-n system, reliability, coherent system, signature, barrier basket option.

1 Introduction

An n-component system that works when at least h of its components work is a *h-out-of-n system.* Such a type of system is widely employed in reliability theory, and has been conceptualized more than fifty years ago (see Birnbaum et al., 1961; Esary and Proschan, 1963, and see also the related extensive treatment of Barlow and Proschan, 1981).
As we will see in details, this paper provides an extension of this concept and applies it to the field of finance, with a specific reference to option theory.

The analysis of the reliability of a h-out-of-n system is the main theme of several contributions in the literature (see e.g. the not recent but very good survey of Chao et al., 1995 and, more recently, Eryilmaz, 2011, 2012, 2013; Freixas and Puente, 2009; Gurler and Bairamov, 2009; Franko and Tutuncu, 2016).

Some variants have been proposed to let the original h-out-of-n system be more tailored on specific reliability issues. The scientific ground of the most part of them lies in the need – for practical applications – of assigning different relevance to the components of the systems when computing their reliability (see e.g. Borgonovo, 2010 for a discussion on this point). Among them, a relevant role is played by the so-called *weighted h-out-of-n systems*, introduced in Wu and Chen (1994). In such models, all the elements of the system are endowed with normalized weights which capture the entity of the contribution of each specific component in assessing the reliability (see e.g. Li and Zuo, 2008; Wang et al., 2012; Faghih-Roohi et al., 2014). A further noticeable extension of the original setting is obtained by assuming that the system works when the elements of some special sets of connected components work (see e.g. Yamamoto et al., 2008).

An equivalent definition of the h-out-of-n systems can be given by replacing "working" with "failed". Indeed, a h-out-of-n system is nothing but a system which fails when at least $n-h+1$ of its components fail. Sometimes, the original h-out-of-n systems are denoted as h-out-of-$n : G$ systems, while the equivalent ones with failures as $n-h+1$-out-of-$n : F$ systems. The letters G and F stand for "Good" and "Failed", respectively.

In this paper we are interested in the events of failure. Thus, to avoid a cumbersome notation, we set $k = n-h+1$ and denote the $n-h+1$-out-of-$n : F$ systems simply as k-out-of-n systems.

As preannounced above, we propose an extension of the concept of k-out-of-n system which includes, in particular, the frameworks discussed in the papers mentioned above. Specifically, we assume that the system fails when, jointly, we have that (i) k components fail; (ii) "some" of the failed components belong to a predefined special set with cardinality $r \in \{0, 1, \dots, n\}$. In our setting, the number of the components of the special set needed for having the failure of the system depends on k, and will be identified through a function ρ as $\rho(k)$ (see the details in the next section). The resulting system will be denoted as *$k/\rho(k)$-out-of-n/r system*.

The reliability functions of the considered systems are explored under the assumption of i.i.d. components, which is natural in this context. Moreover, and importantly, we employ our reliability arguments to construct a financial model for option theory based on the proposed extension of the k-out-of-n systems. In so doing, we derive some intuitive financial results in a very natural way.

Let us enter the details of the financial setting.

Option theory represents one of the most studied topics in finance. Since the beginning of option pricing formalization with the celebrated paper of Black and Scholes (1973), a long history has been developed by academicians as well as practitioners. Options have been classified into several families, starting from the simplest contracts (plain vanilla) to products with a high level of complexity (exotic options). For a survey on option theory, we refer to Huang and Litzenberger (1988) and Hull (2006).

Among all the existing typologies of options, we will concentrate attention on the basket ones. The latter are experiencing an increasing popularity among the retail investors for their constitutive features. Indeed, basket options are written on a set (basket) of assets which can be properly selected by the investors on the basis of their risk-profiles and returns. The payoff of a basket option – and, consequently, its price – is then strongly dependent on the composition of the basket (see e.g. Curran, 1994; Milevsky and Posner, 1998; Brigo et al., 2004; Deelstra et al., 2004; Hobson et al., 2005; Abrahams et al., 2006; Wu et al., 2009; Xu and Zheng, 2009 and 2010; Dingec and Hörmann, 2013; Sesana et al., 2014).

Barrier basket options represent a subclass of the general basket options. In this case, the payoff of the option is linked to the crossing of prefixed thresholds by the returns of the assets in the basket (see Zhang, 1997; Lin, 1998; Brockman and Turtle, 2003; Sun et al., 2007; Dai et al., 2010; Fourati, 2012; Peña et al., 2012; Kim et al., 2015). Once the basket is fixed, the selection of the thresholds leads to the identification of the risk profile and of the expected return of the corresponding option. Thus, the mechanism of thresholds-selection contributes to control for the riskiness of the option.

In this paper, we adopt the extension of k-out-of-n systems mentioned above for defining a special subfamily of barrier basket options, where the components of each option-system are given by the assets forming the basket. Furthermore, options are viewed here as coherent systems (see e.g. Barlow and Proschan, 1981). As we will see below, this assumption fits the financial evidence.

The reliability function of the system will be defined on the basis of the probability that the payoff of the option is positive at a given date (not necessarily the expiration one but, obviously, not after it). The computation of the reliability function is implemented by employing the notion of signature of the system, which has been introduced by Samaniego (1985) and extensively studied in the context of coherent systems by Kochar et al. (1999), Boland and Samaniego (2004), Samaniego (2007), Spizzichino (2008), Navarro et al. (2010), Gertsbakh and Shpungin (2010), Marichal and Mathonet (2011), Marichal et al. (2015) and references therein.

To the best of our knowledge, the concept of signature has been used so far only in the frame of reliability theory. However, we claim that such a notion may be efficiently employed also in the analysis of systems in other fields. In particular, signature can play a useful role in option theory (see Cerqueti and Spizzichino, 2014 for a preliminary analysis).

The barrier basket option model here developed allows us to obtain intuitive financial evidence by using the tools of the reliability field. In particular, we propose a comparison of the risk profiles of different options by comparing the reliability functions of the associated coherent systems. At this aim, we employ some important results of reliability theory on the stochastic orders of the systems

through the concept of signature (Kochar et al., 1999).
The remaining part of the paper is organized as follows. In Section 2 we develop an extension of the standard k-out-of-n models and explore its reliability function by computing its signature. Section 3 contains the barrier basket option model derived by adapting the reliability framework we deal with. Some illustrative examples are also given to assist the reader in catching the links between finance and reliability theory. In Section 4 we implement the comparison between the risk profiles of different barrier basket options in terms of their reliability functions, and provide also some comments on the obtained results. Last section concludes.

2 Preliminaries and notation

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which all the random variables that will be introduced in this paper are defined. The expected value operator with respect to $\mathbb{P}$ will be denoted by $\mathbb{E}$.
We introduce a reliability binary system $\mathbf{S}$ whose n components are denoted by $C_1, \ldots, C_n$. The set collecting the components will be denoted by $\mathcal{B}$.
The *state of the system* $\mathbf{S}$ can be 0 or 1. We define it at time $t > 0$ as:

$$Y(t) = \begin{cases} 1 & \text{if the system has not failed in } (0,t], \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

while the *state of the j-th component C_j* at time t is denoted by:

$$Y_j(t) = \begin{cases} 1 & \text{if } \mathrm{C_j} \text{ has not failed in } (0,t], \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The state of the system $\mathbf{S}$ depends on the states of the components $C_1, \ldots, C_n$. Such a dependence is formalized through the function $\phi : \{0,1\}^n \to \{0,1\}$

$$Y(t) = \phi(Y_1(t), \ldots, Y_n(t)). \tag{3}$$

The function ϕ is usually called the *structure function* of $\mathbf{S}$, and it identifies the configurations of the components which cause the failure of the system.
We assume that the structure function ϕ is such that $\mathbf{S}$ is a *coherent system*. Such an assumption is rather natural in the applications, and will be discussed in details in the next Section.
By definition of the structure function of the system, we can introduce the *system lifetime* as:

$$\mathcal{T} := \inf\{t \geq 0 | \phi(Y_1(t), \ldots, Y_n(t)) = 0\}. \tag{4}$$

Analogously, by (18), we can define the n-dimensional *vector of components lifetimes* as $\mathbf{X} = (X_1, \ldots, X_n)$, where

$$X_j = \inf\{t > 0 \,|\, Y_j(t) = 0\} \tag{5}$$

is the *lifetime of the individual j-th component of the system.*

The quantity of major interest in a reliability model is the reliability function, whose evaluation provides information on how probable is the failure of the system at a certain time. Formally, for $t \geq 0$, the *reliability function* of the system at time t is

$$R_S(t) \equiv \mathbb{P}\{\mathcal{T} > t\}. \tag{6}$$

As intuition suggests, the stochastic model assumed for the random vector $\mathbf{X}$ plays an active role for the assessment of the quantity in (6). In reliability theory, it is often assumed that the components are i.i.d.. We take for us this condition, and assume that the following assumption stands in force hereafter.

Assumption 1. *The components of the random vector* $\mathbf{X}$ *are i.i.d., with an absolutely continuous distribution* F.

An explanation of Assumption 1 is needed.

The absolute continuity of F guarantees that the failure of the system is simultaneous to the failure of a component. Indeed, by including the time dimension in the problem, we notice that the system fails after some consecutive failures of its components, i.e.: in correspondence of the k-th failure. Formally, consider $X_{(1)}, ..., X_{(n)}$ the order statistics of $(X_1, ..., X_n)$. By the absolute continuity we have

$$\mathbb{P}\{X_{(1)} \neq ... \neq X_{(n)}\} = 1,$$

so that:

$$\mathbb{P}\{\mathcal{T} = X_{(k)}, \text{ for some } k\} = 1. \tag{7}$$

We move from the absolute continuity of Assumption 1 and consider the partition $\{E_k\}_{k=1,\ldots,n}$, with

$$E_k \equiv \{\mathcal{T} = X_{(k)}\}, \qquad k = 1, \ldots, n, \tag{8}$$

so that:

$$\mathcal{T} = \sum_{k=1}^{n} X_{(k)} \mathbf{1}_{E_k}. \tag{9}$$

Hence:

$$R_S(t) = \sum_{k=1}^{n} p_k \cdot \mathbb{P}\{X_{(k)} > t | E_k\}, \tag{10}$$

where $p_k = \mathbb{P}(E_k)$, for each $k = 1, \ldots, n$.

The vector $\mathbf{p} = (p_1, \ldots, p_n)$ is the *signature of the system* $\mathbf{S}$ (see Samaniego, 1985), and depends on the structure function of $\mathbf{S}$.

The vector $\mathbf{p}$ plays a relevant role in identifying the reliability function of the system. Indeed, under Assumption 1 – and, more than this, when the components' lifetimes are exchangeable – then it is easy to show (see e.g. Spizzichino, 2008) that

(i) the events $X_{(k)} > t$ and E_k are independent;

(ii) it results:

$$R_S(t) = \sum_{k=1}^{n} p_k \cdot \mathbb{P}\{X_{(k)} > t\}, \tag{11}$$

Thus, in the i.i.d. case, the signature, along with the distribution of the lifetimes of the components, provides full information on the reliability function of the system.

3 The $k/\rho(k)$-out-of-n/r reliability systems

In the context described above, we now extend the usual k-out-of-n models and consider the case where there exists a distinction among the components of the system.

In particular, we assume that the n components can be clustered in two non-overlapping categories: r components are the "important" ones while the remaining $s = n - r$ are "standard" components. The set of the former components will be denoted by $\mathcal{I}$, while $\mathcal{N}$ denotes the complementary set formed with the latter ones, so that

$$\mathcal{I} \cap \mathcal{N} \equiv \emptyset, \qquad \mathcal{I} \cup \mathcal{N} \equiv \mathcal{B}.$$

Conventionally, we label the components in such a way that $\mathcal{I} = \{C_1, \dots, C_r\}$. The sets $\mathcal{I}$ and $\mathcal{N}$ allow us to identify different types of reliability systems.

Specifically, define a nonincreasing function

$$\rho : \{1, \dots, n\} \to \{0, 1, \dots, r+1\}, \tag{12}$$

such that the system fails at the first time in which the failures of k components are observed, with at least $\rho(k)$ failures due to the important components. The resulting reliability model represents an effective extension of the usual k-out-of-n systems. In fact, the proposed setting collapses in the k-out-of-n system when $r = 0$ and $\rho(k) = 0$, for each k or, alternatively, $r = n$ and $\rho(k) = k$, for each k.

We denote such an extended model as *$k/\rho(k)$-out-of-n/r system.*

For $\mathcal{I}$ and $\mathcal{N}$ given, different choices of the function ρ give rise to different types of reliability systems. In any case, the function ρ has the following meaning:

- when $\rho(k) = 0$, then the failure of k components belonging to $\mathcal{N}$ is enough to determine the failure of the system;
- the position $\rho(k) = r + 1$ means that k is so small that the failure of k components cannot produce the system's failure, even in the case when all the failed components are in $\mathcal{I}$;
- the minimum number of components' failures able to potentially cause the failure of the system is the minimum value of k satisfying the condition $\rho(k) \leq k$;

- the maximum possible number of components' failures that can be conceptually observed up to the system's failure does coincide with the minimum value of k such that $\rho(k) = 0$.

Generally, the structure function of the system depends on how the function ρ is defined, and we can write:

$$\phi(y_1, \dots, y_n) = \begin{cases} 0, & \text{if } \rho\left(n - \sum_{j=1}^{n} y_j\right) \leq r - \sum_{j=1}^{r} y_j; \\ \\ 1, & \text{if } \rho\left(n - \sum_{j=1}^{n} y_j\right) > r - \sum_{j=1}^{r} y_j. \end{cases} \tag{13}$$

In fact, the system fails as soon as the following relations are simultaneously true:

$$\begin{cases} \sum_{j=1}^{n} y_j = n - k; \\ \\ \sum_{j=1}^{r} y_j \leq r - \rho(k), \end{cases} \tag{14}$$

namely (14) is equivalent to the first condition in (13).

We are now ready to compute the reliability function of the $k/\rho(k)$-out-of-n/r system.

We compute the signature of the system.

Fix $k = 1, \dots, n$. It is simple to check that, for any function ρ, we have:

- $$p_k = \begin{cases} 0, & \text{if } k = 1 \text{ and } \rho(1) = r + 1; \\ r/n, & \text{if } k = 1 \text{ and } \rho(1) = 1. \end{cases}$$

- $p_k = 0$ if $k > 1$ and $\rho(k) = r + 1$;

- $p_k = 0$ if $k > 1$ and $\rho(k) = \rho(k-1) = 0$;

- $p_k = 1 - \sum_{h \neq k} p_h$ if $k > 1$ and $\rho(k) = 0, \rho(k-1) > 0$.

Let us consider now the general case of $k = 2, \dots, n$ such that $0 < \rho(k) \leq k$. We introduce

$$\bar{s}_k = \mathbb{P}\left(r - \sum_{i \in \mathcal{I}} Y_i(T) \leq \rho(k), \sum_{i=1}^{n} Y_i(T) = n - k\right), \tag{15}$$

which represents the probability of having no more than $\rho(k)$ failures of important components when k components fail.

Assumption 1 guarantees that

$$\bar{s}_k = \sum_{\ell=0}^{\rho(k)-1} \frac{\binom{r}{\ell}\binom{n-r}{k-\ell}}{\binom{n}{k}} \tag{16}$$

Therefore, the probability of exactly $\rho(k)$ failures of important components when k components had failed – i.e.: the k-th component of the signature of the system $\mathbf{S}$ – is given by:

$$p_k = \bar{s}_{k-1} - \bar{s}_k = \sum_{\ell=0}^{\rho(k-1)-1} \frac{\binom{r}{\ell}\binom{n-r}{k-1-\ell}}{\binom{n}{k-1}} - \sum_{\ell=0}^{\rho(k)-1} \frac{\binom{r}{\ell}\binom{n-r}{k-\ell}}{\binom{n}{k}}. \tag{17}$$

4 A barrier basket option model

This section is devoted to the development of an option model in the light of the $k/\rho(k)$-out-of-n/r systems introduced above. When possible, we maintain the same notation adopted in the previous section.

Think of a financial market that contains n assets with stochastic returns at time t given by $\Lambda_1(t), \dots, \Lambda_n(t)$, for each $t \in [0, +\infty)$.

Our interest focuses on a special barrier basket option written on these assets and defined as follows:

> *The holder can exercise the option only at the expiration date T. The return of the option at time T, denoted by $\Pi(T)$, is positive if and only if "some" of the n assets have maintained their own returns greater than predefined positive barriers (knock-out option) in the whole period $(0, T]$. In this case, the return of the option at time T is given by the average of the returns of such assets at time T. If the return of the option is not positive at time T, then the option is said to be failed.*

For each $j = 1, \dots, n$, the barrier associated to the j-th asset are denoted by $\alpha_j \in (0, +\infty)$.

By construction, it is self-evident that the failure of the option can occur before the expiration date T. In fact, if at a given time $t < T$ the returns of "some" of the assets are below the barriers α's, then the option has failed, since the return of the option cannot be positive at time T. In this case, we say that the option has failed in the time period $(0, t]$.

The meaning of the term "some" will be clarified below and, as we will see, it involves the definition of the function ρ in (12).

The barrier basket option is modeled here as a reliability system $\mathbf{S}$ whose n components $C_1, \dots, C_n$ are the assets in the basket $\mathcal{B}$.

The state of the option at time $t > 0$ is $Y(t)$ as in (1), and it depends on the returns of the assets of the basket at t. Actually, such returns can be above or below the barriers α's. In particular, for each $t \in (0, +\infty)$ and $j = 1, \dots, n$, the comparison between the barrier α_j and the return $\Lambda_j(t)$ determines the state of the j-th component C_j at time t, so that we define the state of the j-th asset at time $t > 0$ as the binary variable

$$Y_j(t) = \begin{cases} 1 & \text{if } \Lambda_j(t) > \alpha_j, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

The structure function $\phi : \{0,1\}^n \to \{0,1\}$ in (3) formally describes the dependence between the state of the option and the ones of its components.

As preannounced in the previous Section, it is reasonable to require that ϕ is such that option $\mathbf{S}$ is a coherent system, i.e.:

(*i*) $\phi(0, \dots, 0) = 0$ and $\phi(1, \dots, 1) = 1$.

This condition means that when all the assets have returns not above the thresholds α's, then the payoff of the option is not positive; differently, when the j-th return is greater than α_j, for each $j = 1, \dots, n$, then the option has positive payoff.

(ii) ϕ is non-decreasing with respect to its components.

This requirement expresses that the failure of one of the assets of the basket cannot lead to an improvement of the state of the option.

(iii) Each component of the system is "relevant".

Formally, this condition states that it does not exist $j = 1, \dots, n$ such that

$$\phi(y_1, \dots, y_{j-1}, 0, y_{j+1}, \dots, y_n) = \phi(y_1, \dots, y_{j-1}, 1, y_{j+1}, \dots, y_n),$$

for each $(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n) \in \{0, 1\}^{n-1}$.

It is worth noting that, by construction, the asset C_j $(1 \le j \le n)$ is capable in principle to survive at any t, even if $\mathbf{S}$ has already failed in $(0, t]$. According to formulas (4) and (5), the lifetimes of the option and of the j-th asset will be denoted by $\mathcal{T}$ and X_j, respectively, for each $j = 1, \dots, n$. Needless to say, Assumption 1 is true also for this collection of X's.
For the convenience of the reader, we now present an illustrative example.

Example 1. *Consider $n = 5$ assets $C_1, \dots, C_5$, positive thresholds (barriers) $\alpha_1, \dots, \alpha_5$, and suppose that the barrier basket option fails in one of the following three cases:*

$(c1)$ *In the period $(0, T]$, the returns of the assets C_1, C_3 and C_4 go below the thresholds α_1, α_3 and α_4, respectively;*

$(c2)$ *In the period $(0, T]$, the returns of the assets C_2 and C_5 go below the thresholds α_2 and α_5, respectively.*

The structure function can be constructed on the basis of the expiration conditions $(c1)$, $(c2)$ and $(c3)$ as follows:

$$Y(T) = \phi(Y_1(T), \dots, Y_5(T)) = \begin{cases} 0, & \textit{if } (Y_1(T), Y_3(T), Y_4(T)) = (0, 0, 0) \textit{ or } (Y_2(T), Y_5(T)) = (0, 0); \\ 1, & \textit{otherwise.} \end{cases} \tag{19}$$

It is easy to check that the barrier basket option in this example – with the structure function ϕ defined as in (19) – is a coherent reliability system.

Example 1 sheds some light also on the meaning of the term "some" in the definition of the barrier basket option. The failure of the option as a reliability system is, indeed, caused by the joint failure of some specific set of components over the period $(0, T]$.
In particular, we assume that the barrier basket option is a $k/\rho(k)$-out-of-n/r reliability system. Specifically, the basket $\mathcal{B}$ is clustered into a set of important assets $\mathcal{I} = \{C_1, \dots, C_r\}$ and the set of

standard ones $\mathcal{N} = \{C_{r+1}, \ldots, C_n\}$. The failure of the option is then explained by the identification of a function ρ as in (12).

We now demonstrate the financial meaning of the function ρ through an illustrative example.

Example 2. *Consider $n = 20$ assets $C_1, \ldots, C_{20}$ along with their positive thresholds (barriers) $\alpha_1, \ldots, \alpha_{20}$. Suppose that the barrier basket option fails in the following cases:*

($d1$) *the returns of five assets – including at least one among C_1, C_2, C_3 and C_4 – go below their reference thresholds in the period $(0, T]$;*

($d2$) *the returns of four assets – including at least two among C_1, C_2, C_3 and C_4 – go below their reference thresholds in the period $(0, T]$;*

($d3$) *the returns of three assets selected among C_1, C_2, C_3 and C_4 go below their reference thresholds in the period $(0, T]$;*

($d4$) *more than five assets have return going below their reference thresholds in $(0, T]$.*

Thus, it is easy to identify $r = 4$ important assets, namely $\mathcal{I} \equiv \{C_1, C_2, C_3, C_4\}$. The remaining $n - r = 16$ assets are standard ones.

The conditions $(d1) - (d4)$ lead us to define the function ρ as follows:

$$\rho(k) = \begin{cases} 0, & \text{if } k \geq 6; \\ 1, & \text{if } k = 5; \\ 2, & \text{if } k = 4; \\ 3, & \text{if } k = 3; \\ 5, & \text{if } k < 3. \end{cases} \tag{20}$$

The structure function can be constructed in a natural way on the basis of the function ρ, according to (13) and (20).

In the specific framework we are considering, the function R_S in (6) provides an appropriate measure of the riskiness of the option. In particular, it is important to consider the value of the function R_S at the expiration date T, i.e. $R_S(T)$. In fact, the financial contract we deal with considers the situation of the assets' returns at the expiration date in order to establish the final payoff of the option. Thus, a correct evaluation of $R_S(T)$ is a crucial step for the determination of the risk profile associated to the option $\mathbf{S}$.

Moreover, Assumption 1 has a specific financial motivation. In fact, this condition means that the basket is composed by elements, which are homogeneous as to their probabilities to fail. However, since the thresholds α's have in general different values, the i.i.d. condition does not exclude that two different elements of the basket might have different expected returns and volatilities. Now, recall that the return of the option at the expiration date T – if positive – is given by the average of the returns of the single assets above the barriers α's at time T. Therefore, the heterogeneity of the assets of the basket leads to multiple opportunities for the return at T of the barrier basket option.

5 Comparison between barrier basket options

In this Section we establish a comparison between barrier basket options and obtain an interpretation in our context of intuitive financial results.

The choice of the sets $\mathcal{I}$ and $\mathcal{N}$ plays a key role in determining the returns of such financial products. We now need the following:

Notation 1. *Consider two barrier basket options* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *whose returns at time* T *are* $\Pi_1(T)$ *and* $\Pi_2(T)$*, respectively. For* $j = 1, 2$*,* $\mathbf{S}_j$ *has basket* $\mathbf{B}_j$*, set of important assets* $\mathcal{I}_j$ *and set of standard ones* $\mathcal{N}_j$*. Accordingly, the function* ρ *of the option* j *will be denoted by* ρ_j*.*

The k*-th asset lifetime of option* $\mathbf{S}_j$ *will be denoted by* $X_k^{(j)}$*, for* $k = 1, \ldots, n$ *and* $j = 1, 2$*.*

The generic $\bullet$*-th element of* $\mathbf{B}_j$ *will be denoted by* $C_\bullet^{(j)}$*, whose return at time* T *is* $\Lambda_\bullet^{(j)}(T)$*, for* $j = 1, 2$*.*

Furthermore, we state the following set of assumptions:

Assumption 2. *The baskets* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are disjoint sets with the same cardinality* n*, namely* $|\mathcal{B}_1| = |\mathcal{B}_2| = n$*.*

The assets' lifetimes $X_1^{(1)}, \ldots, X_n^{(1)}$ *and* $X_1^{(2)}, \ldots, X_n^{(2)}$ *are i.i.d..*

The sets $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *share the same cardinality* r*, namely* $|\mathcal{I}_1| = |\mathcal{I}_2| = r$*.*

Finally, $\rho_1 \equiv \rho_2$*, and we set* $\rho = \rho_1 = \rho_2$*.*

We now state two results which mark the difference between standard and important assets, once they are compared.

Proposition 1. *Consider two barrier basket options* $\mathbf{S}_1$ *and* $\mathbf{S}_2$*. Suppose that Assumption 2 is satisfied.*

Moreover, suppose that:

$(H1.1)$ *For any pair* $(C^{(1)}, C^{(2)}) \in \mathcal{I}_1 \times \mathcal{I}_2$*, one has* $\mathbb{P}(\Lambda^{(1)}(T) > \Lambda^{(2)}(T)) = 1$*.*

$(H1.2)$ *For all* $(C^{(1)}, C^{(2)}) \in \mathcal{N}_1 \times \mathcal{N}_2$*, one has* $\mathbb{P}(\Lambda^{(1)}(T) = \Lambda^{(2)}(T)) = 1$*.*

$(H1.3)$ $\rho(k) > 0$*, for each* $k = 1, \ldots, n$*, and there exists* $\bar{k} \in \{1, \ldots, n\}$ *such that* $\rho(\bar{k}) = \bar{k}$ *and* $\rho(\bar{k} - 1) = r + 1$*.*

Suppose also that the options share the same number k_r *of failed important assets and* k_n *of standard ones in* $(0, T]$*, and the failures are such that the options have not failed in* $(0, T]$*.*

Then $\mathbb{P}(\Pi_1(T) > \Pi_2(T)) = 1$*.*

Proof. First of all, we notice that Assumption 2 gives that also the sets of the standard assets, denoted by $\mathcal{N}_1$ for $\mathbf{S}_1$ and $\mathcal{N}_2$ for $\mathbf{S}_2$, have the same cardinality: $|\mathcal{N}_1| = |\mathcal{N}_2| = n - r$.
Second of all, hypothesis $(H1.3)$ and the condition of not failure of the options guarantee that $k \geq \bar{k}$, where $k = k_r + k_n$.
Now, fix $j = 1, 2$ and consider the barrier basket option $\mathbf{S}_j$. Denote by $\mathcal{O}_{\mathcal{I}_j} \subseteq \mathcal{I}_j$ and $\mathcal{O}_{\mathcal{N}_j} \subseteq \mathcal{N}_j$ the sets of the not failed assets in $(0, T]$ of important and standard type, respectively.
By hypothesis, the return of the j-th option at time T is positive, and it is given by

$$\Pi_j(T) = \frac{1}{k} \left[\sum_{i \in \mathcal{O}_{\mathcal{I}_j}} \Lambda^{(i)}(T) + \sum_{h \in \mathcal{O}_{\mathcal{N}_j}} \Lambda^{(h)}(T) \right].$$

The possible realizations of the return of the j-th option at time T are given by the possible combinations of the not failed assets. Therefore, we need to define the set collecting the possible selections of x elements in a set of y elements, being x and y integers. We denote this set by $\mathcal{Q}_x^{(y)}$ and, of course, $|\mathcal{Q}_x^{(y)}| = \binom{x}{y}$.
So, define the set

$$\mathcal{A}(k, k_r, k_n) = \mathcal{Q}_{k_r}^{(r)} \times \mathcal{Q}_{k_n}^{(n-r)},$$

for $k \in \{\bar{k}, \ldots, n\}$, $k_r \in \{\rho(k), \ldots, r\}$ and $k_n = k - k_r$. Then

$$\Pi_j(T) \in \bigcup_{k, k_r, k_n} \bigcup_{a \in \mathcal{A}(k, k_r, k_n)} \{\Pi_j^{(a)}(T)\},$$

where

$$\Pi_j^{(a)}(T) = \frac{1}{k} \left[\sum_{i \in \{i_1, \ldots, i_{k_r}\}} \Lambda_i^{(j)}(T) + \sum_{h \in \{h_1, \ldots, h_{k_n}\}} \Lambda_h^{(j)}(T) \right], \tag{21}$$

for each $a = (\{i_1, \ldots, i_{k_r}\}, \{h_1, \ldots, h_{k_n}\}) \in \mathcal{A}(k, k_r, k_n)$.
Assumptions $(H1.1)$, $(H1.2)$ and $(H1.3)$ and formula (21) assure that

$$\mathbb{P}\left(\Pi_1^{(a_1)}(T) > \Pi_2^{(a_2)}(T)\right) = 1, \tag{22}$$

for each $a_1, a_2 \in \mathcal{A}(k, k_r, k_n)$, for each k, k_r, k_n.
Now define the event

$$\Gamma_a^{(j)} = \left\{ C_i^{(j)}, C_h^{(j)} \text{ have not failed in } (0, T] \text{ iff } (i, h) \in \mathcal{A}(k, k_r, k_n) \right\}, \qquad j = 1, 2.$$

By the Total Probability Theorem one has

$$\mathbb{P}\left(\Pi_1(T) > \Pi_2(T)\right) = \mathbb{P}\left(\bigcap_{k, k_r, k_n} \bigcap_{a_1, a_2 \in \mathcal{A}(k, k_r, k_n)} \{\Pi_1^{(a_1)}(T) > \Pi_2^{(a_2)}(T)\} \cap \Gamma_{a_1}^{(1)} \cap \Gamma_{a_2}^{(2)} \right) = 1, \tag{23}$$

and this concludes the proof. $\square$

We now invert the role of standard and important assets as in Proposition 1, to give a distinction between the two typologies of components of the basket of an option.

Proposition 2. *Consider two barrier basket options* $\mathbf{S}_1$ *and* $\mathbf{S}_2$*. Suppose that Assumption 2 is satisfied.*

Moreover, suppose that:

($H2.1$) *For any pair* $(C^{(1)}, C^{(2)}) \in \mathcal{I}_1 \times \mathcal{I}_2$*, one has* $\mathbb{P}(\Lambda^{(1)}(T) = \Lambda^{(2)}(T)) = 1$*.*

($H2.2$) *For all* $(C^{(1)}, C^{(2)}) \in \mathcal{N}_1 \times \mathcal{N}_2$*, one has* $\mathbb{P}(\Lambda^{(1)}(T) > \Lambda^{(2)}(T)) = 1$*.*

($H1.3$) $\rho(k) > 0$*, for each* $k = 1, \dots, n$*, and there exists* $\bar{k} \in \{1, \dots, n\}$ *such that* $\rho(\bar{k}) = \bar{k}$ *and* $\rho(\bar{k} - 1) = r + 1$*.*

Suppose also that the options share the same number k_r *of failed important assets and* k_n *of standard ones in* $(0, T]$*, and the failures are such that the options have not failed in* $(0, T]$*.*

Then $\mathbb{P}(\Pi_1(T) > \Pi_2(T)) < 1$*.*

Proof. We adapt here some of the parts of the proof of Proposition 1 and, when possible, the notation.

Consider $k = \bar{k}$ and $k_r = \bar{k}$, $k_n = 0$. Then $\mathcal{O}_{\mathcal{N}_j} = \emptyset$, and in this specific case

$$\Pi_j(T) = \frac{1}{\bar{k}} \left[\sum_{i \in \mathcal{O}_{\mathcal{I}_j}} \Lambda^{(i)}(T) \right]. \tag{24}$$

Hypothesis ($H2.1$) and (24) guarantee that

$$\mathbb{P}\left(\Pi_1(T) = \Pi_2(T)\right) > 0, \tag{25}$$

hence giving the thesis. □

Propositions 1 and 2 provide a formal view of the different role played by the standard and important assets in determining the failure of the option and its final return. At this aim, a suitable selection of function ρ – as in hypothesis ($H1.3$) – is needed.

In fact, as already explained above, $\rho(k) = 0$ implies that the failure of k assets – taken indifferently from the important and the standard ones – is enough to determine the failure of the option. Thus, in this case, one cannot appreciate the difference between standard and important assets: all of them play an identical role in determining the failure of the option.

Furthermore, the existence of $\bar{k}$ such that $\rho(\bar{k}) = \bar{k}$ and $\rho(\bar{k} - 1) = r + 1$ assures that – at least in some cases – the failure of all important assets is required to have the failure of the option. If such a condition is violated, then $\rho(k) \neq k$ for each $k = 1, \dots, n$. In this case, we are in the position of defining a new function

$$\tilde{\rho} : \{1, \dots, n\} \to \{1, \dots, n - r + 1\}, \tag{26}$$

such that the option fails at the first time in which the failures of k assets are observed, with no more than $\tilde{\rho}(k)$ failures due to the standard assets. The position $\tilde{\rho}(k) = n - r + 1$ means that k

failures are not enough to determine the failure of the option, so that

$$\rho(k) = r + 1 \qquad \text{iff} \qquad \tilde{\rho}(k) = n - r + 1.$$

Otherwise, $\rho(k)+\tilde{\rho}(k) = k$, for each $k = 1, \ldots, n$, and $\tilde{\rho}(k) > 0$. In this case, the thesis of Proposition 2 might be no longer true. In fact, it is possible to produce examples such that the positive final return of the option is due also to some standard assets, hence invalidating a writing like (24). As a consequence, (25) is not longer true.

As a consequence, the role of standard and important assets can be well illustrated only by assuming hypothesis $(H1.3)$. In this respect, a side result of Propositions 1 and 2 is also the interpretation of function ρ.

We now present a comparison of the risk profiles of two barrier basket options by applying the concept of signature. We work under Notation 1 and the following assumption:

Assumption 3. *The barrier basket options* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *share a common basket, i.e.:* $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B} = \{C_1, \ldots, C_n\}$. *Moreover,* $\rho_1 \equiv \rho_2$, *and we denote* $\rho = \rho_1 = \rho_2$.

The reliability of a system – i.e. the risk profile of a barrier basket option – comes out from combining the information on the number of important assets and on the action of function ρ. To highlight this aspect of the financial-reliability model, we present two results.

Proposition 3. *Consider two barrier basket options* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *and suppose that Assumption 3 is satisfied. Suppose that:*

$(H2.1)$ $|\mathcal{I}_1| = r_1 \leq r_2 = |\mathcal{I}_2|$;

$(H2.2)$ *function* ρ *is defined as follows:*

$$\rho(k) = \begin{cases} 0, & \text{for } k > 1; \\ 1, & \text{for } k = 1. \end{cases}$$

Then it results:

$$R_{S_1}(t) \geq R_{S_2}(t), \qquad \forall\, t \geq 0.$$

Proof. For $j = 1, 2$, the signature of the system $\mathbf{S}_j$ is:

$$p_k^{(j)} = \begin{cases} r_j/n, & \text{for } k = 1; \\ 1 - r_j/n, & \text{for } k = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we obtain:

$$\begin{cases} \sum_{k=i}^{n} p_k^{(1)} = \sum_{k=i}^{n} p_k^{(2)}, & \text{for } i \neq 2; \\ \sum_{k=i}^{n} p_k^{(1)} \geq \sum_{k=i}^{n} p_k^{(2)}, & \text{for } i = 2. \end{cases} \tag{27}$$

By applying Kochar et al. (1999) we get $R_{S_1}(t) \geq R_{S_2}(t)$, for each t. □

In the case treated by the Proposition 3, the barrier basket option is assumed to fail at the second failure of the assets of the basket or at the first failure of an important one. In this context, the relative number of important assets is inversely related to the reliability of the system. The theoretical motivation for this outcome lies in the fact that the probability of the failure is: (i) higher for $\mathbf{S}_1$ than for $\mathbf{S}_2$, at the failure of the second component; (ii) lower for $\mathbf{S}_1$ than for $\mathbf{S}_2$, at the failure at the first asset. The tradeoff between these probabilities gives that case (ii) "dominates" case (i), and $\mathbf{S}_2$ is then the less reliable system.

Next result presents a different setting:

Proposition 4. *Consider two barrier basket options $\mathbf{S}_1$ and $\mathbf{S}_2$ and suppose that Assumption 3 is satisfied. Suppose that:*

$(H3.1)$ $|\mathcal{I}_1| = r_1 \leq r_2 = |\mathcal{I}_2|$, *with* $r_1 > n/2$;

$(H3.2)$ *it results* $\rho(k) = 1$, *for each* $k = \{1, \ldots, n\}$.

Then it results:

$$R_{S_1}(t) \geq R_{S_2}(t), \qquad \forall\, t \geq 0.$$

Proof. Define as $(p_1^{(j)}, \ldots, p_n^{(j)})$ the signature of the system $\mathbf{S}_j$, with $j = 1, 2$. By hypotheses $(H3.1)$ and $(H3.2)$, we can rewrite (17) as follows:

$$p_k^{(j)} = \frac{\binom{n-r_j}{k-1}}{\binom{n}{k-1}} - \frac{\binom{n-r_j}{k}}{\binom{n}{k}} = \frac{(n-k)!}{n!} \cdot r_j(n-m_j)(n-r_j-1) \cdot \cdots \cdot (n-r_j-k). \tag{28}$$

Let us pose $x_j = n - r_j$ and define $f(x_j) = (n - x_j)x_j(x_j - 1) \cdot \cdots \cdot (x_j - k)$, so that (29) leads to:

$$p_k^{(j)} = \frac{(n-k)!}{n!} \cdot f(x_j). \tag{29}$$

By hypothesis $(H3.1)$, a simple computation gives that $f'(x_j) > 0$. This implies that $p_k^{(j)}$ decreases with respect to r_j, hence leading to $p_k^{(1)} > p_k^{(2)}$, for each $k = 2, \ldots, n$. Thus, we obtain:

$$\sum_{k=i}^{n} p_k^{(1)} \geq \sum_{k=i}^{n} p_k^{(2)} \qquad i = 1, 2, \ldots, n. \tag{30}$$

Since the options share the same basket and by inequality (30), we are in the position to apply Kochar et al. (1999), who guarantee that $R_{S_1}(t) \geq R_{S_2}(t)$. $\square$

Proposition 4 concerns the case of a barrier basket option failing at the first failure of an important asset. In this context, we have a sufficient condition for having an inverse relation between the number of important assets and the reliability function of the system. Specifically, such condition is that the number of important assets is greater than that of the standard ones. Proposition 4 is rather intuitive. Indeed, the i.i.d. condition in Assumption 1 gives that a large relative number of important assets in the basket would be associated to a more probable failure of an element of $\mathcal{I}$. This suggests a low value for the reliability of the system when the action of function ρ is such that the option fails at the moment of the failure of an important asset.

6 Conclusions and future research

This paper contains an extension of the classical k-out-of-n systems, obtained by assigning different roles to the components of a coherent system in term of the reliability. In particular, the components are clustered in important and standard ones, and the failure of the system depends on how many components of the two sets are failed.

The reliability framework is adapted to the construction of a barrier basket option model, so that the assets of the basket are classified in standard and important ones. In accord to the reliability model, assets' lifetimes are assumed to be i.i.d. Such a condition does not lead to a restrictive setting, and has also the advantage of letting the model be more tractable under a mathematical point of view. The risk profile of the option is explored by the direct computation of the reliability function of the system. Moreover, the different roles of important and standard assets is illustrated and commented. Furthermore, different barrier basket options are compared in terms of their reliability functions. At this aim, the system signature is used for adopting important results on the stochastic order of reliability systems. The comparison provides insights on how the action of the function ρ and the cardinality of the set $\mathcal{I}$ can be combined to identify the reliability of the system.

It would be challenging to further extend the k-out-of-n models by removing the i.i.d. condition for the components of the systems. In this respect, the computation of the signature and the resulting financial model offers a not easy treatment, since the equivalence between formulas (10) and (11) is not longer true. We leave this topic to future research.

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